A Technical Note: The Single-Period Stochastic-Demand Inventory Model under Human Learning

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Abstract- **We consider the inventory model under the environment of the worker learning, single period, and stochastic demand with the research objective of proposing the cost-minimization order policy. Our work differs from the classical Newsvendor Model in that we incorporate the human factor (specifically worker learning) and its influence over the costs of processing units into the model. We found a number of important characteristics related to the expected cost function and its derivatives; we then used them in formulating the optimal ordering policy. Our research results could be helpful for the analysis of the supply chain coordination and the extension of a periodic review inventory system for similar problems.**

Keywords - supply chain management, stochastic inventory model, optimal order policy, worker learning

I. INTRODUCTION

We consider the worker-learning stochastic inventory model, assuming lost-sales for the unmet demand. The lot sizes are assumed to arrive at the beginning of the selling-period. In general, the single-period stochastic inventory model (also known as the Newsvendor Model) is used to find the optimal order quantity for the perishable items such as fashionable products or those with seasonal demand or short-life cycles. Technically, it is used when the demand for product is stochastic and available for the single selling-season, and when there is an only one-time opportunity for the vendor to purchase, with high possibility of experiencing the long delivery lead time. Our work differs from the classical Newsvendor Model in that we incorporate the human learning and its influence over the costs of processing units into the model.

We describe human learning by using the well-known widely-used Wright's Learning Curve [7]. Our problem is challenging in the way that the best order quantity in the classical model, which is balancing the over-stocking and under-stocking costs, is no longer optimal. Specifically, when adding the cost-saving from worker learning on to the expected total cost, the convexity of the cost curve possibly would not exist. This calls for a new way in determining the optimal order policy.

In response to such challenges, we found a number of important characteristics related to the expected cost function and its derivatives, which we then used in formulating the optimal ordering policy. Examples of the optimal characteristics are:

a) The specific levels of the order quantity q exist that satisfy the first order condition;
b) The optimal order quantity exists and is unique if the demand follows a Uniform Distribution;
c) The unique optimal order quantity exists if demand follows the Beta Distribution with some specific properties of its parameters.

The paper is structured as follows:

a) Section II provides literature review for the work related to single-period inventory models with quantity discount programs, price-dependent demand, and/or learning by experience.
b) Section III presents the formulation of the single-period stochastic-demand inventory model, with the incorporation of human learning. The formulation specifies the research objective of the cost minimization, and expresses the expected cost $G(q)$ function in a mathematical formula.
c) Section IV illustrates the properties of the cost function together with their derivatives, and the proofs of the results. The optimal policies are discussed in Section IV.
d) We then provide a concluding remark and the possible future research in Section V.

II. LITERATURE REVIEW

Khouja [4] extended the single-period stochastic inventory model to the case in which demand is price-dependent and multiple discounts with prices are allowed. The discounts are used to sell excess inventory. He developed the algorithm to identify the optimal order quantity, analyze the joint determination of the order quantity and initial price, and provide some insights using numerical examples. Weng [6] generalized Newsvendor Model by analyzing the coordinated quantity decisions between the manufacturer and the buyer. Their analysis developed the insights into the use of quantity discount as a coordination mechanism between the manufacturer and the buyer. Hua, Wang, and Cheng [3] considered the Newsvendor Model in which, given the supplier offers free shipping, the retailer faces stochastic demand; their algorithm was used to determine the retailer's optimal order quantity and the optimal selling price simultaneously. They incorporated the supplier's quantity discount and transportation costs into the models.

Arshavskiy, Okulov and Smirnova [1] conducted three different experiments on the basis of Newsvendor Model. In the first experiment, the optimal decision depends on the uncertain number of buyers; in the second experiment, the optimal decision is depending on the uncertain demand and the competitors' decisions; in the third experiment, the optimal decision is subject to the factors similar to those in the second experiment and on decisions of a participant and his/her competitors.

Bolton and Katok [2] conducted a laboratory study and investigated whether enhancements to experience and feedback can facilitate better newsvendor learning-by-doing. Parts of their hypotheses concern adaptive learning and forward-looking learning. They reported that how experience and feedback are organized for the decision maker may have an important influence on whether inventory is stocked optimally. Their findings are mainly obtained from the laboratory / experimental study. To our best knowledge, none of the existing publications have illustrated the theoretical results in the area of single-period stochastic-demand inventory model in a manufacturing environment in which workers learn and are able to produce more efficiently as they process additional units. We probably are the first to include the worker learning and its influence over the costs of processing units into the Newsvendor Model.

III. MODEL FORMULATION

Various assumptions in the classical single-period stochastic inventory model are held in our model, including one-period stochastic-demand (x) with the pdf $f(x)$ and cdf $F(x)$, only one opportunity of order placement, the lost-sales taken place for the unmet demand, the constant per-unit cost of leftover (ℓ) and shortage (s), as well as the continuous time. Although the objective of this study is to maximize the expected profit of $\Pi(q)$, it is equivalent to minimizing the expected cost of $G(q)$, by selecting the optimal inventory level q^* at the beginning of the period.

Our work incorporates the human-learning, as described by Wright's learning curve [7], with the following specifications:

$$T(y) = T(1)y^{-m},$$

where y is the cumulative production volume, $T(y)$ is the production time for the y^{th} unit, and m is the slope of the learning curve, and $m = -\log(p)/\log(2)$ with $100(1-p)$ be the percentage decline in the unit production time with a doubling of the number of units. Note that we assume the production time is continuous and therefore we use the continuous learning curve; that is, y is allowed to take non-integer values and $T(1)$ is the initial instantaneous per-unit production time at the state with no learning. More details of the continuous learning curve is available in the work of Teyarachakul, Chand and Ward [5].

The total production cost for a batch of q units that starts with no prior leaning is defined by $PC(q)$ function:

$$PC(q) = C_0 \int_1^{1+q} x^{-m}\,dx = \frac{C_0}{1-m}\left[(1+q)^{1-m} - 1\right] \qquad (1),$$

where, C_o is the initial instantaneous per-unit production cost at the state, prior to any learning. With minor modifications, $PC(q)$ can be used to compute the batch production cost of q units, starting with some prior learning of workers.

Profit, Cost and Objective Functions

The model for expected profit $\Pi(q)$ and cost $G(q)$ are illustrated as follows:

$$\Pi(q) = rE(\text{Sales}|q) - \{\frac{C_0}{1-m}\left[(1+q)^{1-m} - 1\right] + \ell\, E(\text{Leftover}|q) + sE(\text{Shortage}|q)\}$$

$$= rE(x) - \{\frac{C_0}{1-m}\left[(1+q)^{1-m} - 1\right]) + \ell\, E(\text{Leftover}|q) + (r+s)E(\text{Shortage}|q)\}$$

where r is the revenue made from each unit sold;

ℓ is the out-of-pocket leftover cost per unit ($\ell \geq 0$) charged at end of period such as the inventory-holding cost, disposal cost, salvage (resale) value, etc.;

s is the shortage cost per unit ($s \geq 0$) charged at end of period such as the opportunity cost of making more profit, loss of goodwill, penalty cost, etc.

If $F(a) = \int_0^a f(x)dx$ and the expected demand $E(x) = \int_0^\infty xf(x)dx$, then $E(\text{Sales}|q) = E(x) - E(\text{Shortage}|q)$.

The objective is to select the optimal amount of the inventory (or order quantity q) that minimizes the expected cost $G(q)$; that is

$$\text{Min } G(q) = \underset{q \geq 0}{\text{Min}} \left\{ \frac{C_0}{1-m}\left[(1+q)^{1-m} - 1\right] + \ell\, E(\text{Leftover}|q) + (r+s)E(\text{Shortage}|q) \right\}$$

$$= \underset{q \geq 0}{\text{Min}} \left\{ \frac{C_0}{1-m}\left[(1+q)^{1-m} - 1\right] + \ell\int_0^q (q-x)f(x)dx + p\int_q^\infty (x-q)dx \right\},$$

where $p = r + s$.

$$G(q) = \frac{C_0}{1-m}\left[(1+q)^{1-m} - 1\right] + \ell\int_0^q (q-x)f(x)dx + p\int_q^\infty (x-q)dx \qquad (2)$$

The derivative of $G(q)$ are specified by

$$G'(q) = C_o(1+q)^{-m} - p + (\ell + p)F(q)\,;$$
$$G''(q) = -mC_o(1+q)^{-m-1} + (\ell + p)f(q).$$

Note that $G''(q)$ may not be positive, and, therefore, $G(q)$ may not exhibit the convexity of the expected cost in the order quantity q.

Let q^* be the optimal inventory-level q at the beginning of the period; so, $G(q^*)$ is the minimal cost of $G(q)$. We denote $\hat{q}$ is the value of q that satisfies $G'(\hat{q}) = 0$. Although there is no guarantee in the convexity of cost $G(q)$ in general, we are able to show that there exists a unique optimal solution q^* when demand follows the distributions of Uniform or Beta with some specific characteristics to be discussed next.

IV. CHARACTERISTICS OF THE EXPECTED COST AND THE OPTIMAL ORDER POLICY

***Characteristic* 1:**

There exists q that satisfies the first order condition.

Proof

Recall $G'(q) = C_o(1+q)^{-m} - p + (\ell + p)F(q)$. Since $\lim_{q\to 0} G'(q) = C_0 - p < 0$ and $\lim_{q\to\infty} G'(q) = \ell > 0$, there must be some values $\hat{q}$, where $0 < \hat{q} < \infty$, such that $G'(\hat{q}) = 0$.

Note that we cannot rule out the possibility that $\hat{q}$ may not be unique. By the first order condition, $\frac{1}{\ell + p}(p - C_0(1+q)^{-m}) = F(q)$. Both LHS and RHS of the equation are increasing as q increases. There are possibilities that they are equal to each other more than once.

***Characteristic* 2:**

q^* is unique if demand x is uniformly distributed.

Proof

We will first show that if demand x follows the Uniform Distribution, $G''(q)$ is either positive (and $G(q)$ is convex) or has one root. Then we will draw the conclusion that the optimal q^* is unique.
Recall the Uniform pdf $f(x) = \frac{1}{b-a}$ and cdf $F(q) = \frac{q-a}{b-a}$. Let b be very large x and $b < \infty$. Then,

$$G''(q) = -mC_o(1+q)^{-m-1} + (\ell + p)\frac{1}{b-1}$$

If $mC_o(1+q)^{-m-1} \leq (\ell + p)\frac{1}{b-1}$, $G''(q) \geq 0$ and so $G(q)$ is convex and q^* is unique.

If $mC_o(1+q)^{-m-1} > (\ell + p)\frac{1}{b-1}$, $G''(q)$ has one root. Although $G(q)$ may not be convex, q^* is unique. See Figure 1 in Appendix *A* for illustration. Following from Characteristic 2, there are at most two values of $\hat{q}$; this justifies the Optimal Policy 1U.

Optimal Policy 1U: For the uniform distributed demand, the optimal order quantity q^* is such that

$$q^* = \underset{\forall(1,\hat{q})}{\arg\min}\, G(q),$$

where $\hat{q}$ satisfies $C_o(1+\hat{q})^{-m} - p + (\ell + p)\frac{\hat{q}-a}{b-a} = 0$

Next we consider the case when demand x follows the Beta Distribution.

***Characteristic* 3:**

q^* is unique if demand x follows the Beta Distribution with $\alpha = \beta$ and $\beta \in \{2, 4, 6, \ldots\}$.

Proof

Beta pdf $f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, where $B(\alpha,\beta) = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$ is a constant; $\alpha > 0$; $\beta > 0$; and $x \in [0, 1]$.

In our model, the possible demand $x \in (0, \infty)$, we need to truncate the demand range to fit with the requirement of the Beta Distribution of $x \in [0, 1]$, by letting new x = old x / $Maxq$, where $Maxq$ is an arbitrarily very large demand such that any production beyond Maxq gives a much too high overstocking cost, far higher than understocking cost. At the root of $G''(q)$, $mC_o(1+q)^{-m-1} = (\ell + p)f(q)$

$$mC_o(1+q)^{-m-1} = (\ell + p)c\left(\frac{q}{Maxq}\right)^{\alpha-1}\left(1-\frac{q}{Maxq}\right)^{\beta-1}.$$

A constant c is used to adjust value of $(\ell + p)$ due to newly defined demand x; c also takes into account the effect of $\frac{1}{B(\alpha,\beta)}$. As a result,

$$mC_o(1+q)^{-m-1} = (\ell + p)c\, Maxq^{2-\alpha-\beta} q^{\alpha-1}(Maxq - q)^{\beta-1}.$$

If $\alpha = \beta$ and $\beta \in \{2, 4, 6, \ldots\}$, then $q^{\alpha-1}(Maxq - q)^{\beta-1}$ or RHS is increasing convex and LHS is decreasing convex. Therefore $G''(q)$ has one root and q^* is unique. Figure of $G''(q)$, $G'(q)$, and $G(q)$ for the Beta distributed demand is similar to that for the uniform distributed demand. See Appendix *A*.

Optimal Policy 1B: For the Beta distributed demand with $\alpha = \beta$ and $\beta \in \{2, 4, 6, \ldots\}$, the procedure to obtain the optimal q^* is the following:

1. Find q' from

$$mC_o(1+q')^{-m-1} = (\ell + p)c\, Maxq^{2-\alpha-\beta} q'^{\alpha-1}(Maxq - q')^{\beta-1}.$$

2. Find the value of

$$G'(q') = C_o(1+q')^{-m} - p + (\ell + p)F(q'/Maxq)$$

3. If the value of $G'(q') = 0$, then $q^* = 1$. Otherwise, use the bisection method as suggested to find the optimal q^* where $q^* \in (q', Maxq)$

$$q^* = \underset{\forall q \in (q', Maxq)}{\arg\min}\ G(q)$$

We will close our presentation by providing the concluding remarks.

V. CONCLUDING REMARK

We extend the single-period inventory model by incorporating worker learning in processing units into the classical model. Most of the assumptions of the classical stochastic inventory model are still maintained in our work, such as the constant per-unit cost of leftover and shortage, the zero initial inventory, as well as the continuous time. Wright's Learning Curve [7] is used to explain learning in processing units. We found a number of important characteristics related to the expected cost function and its derivatives and suggested the optimal ordering policy when the demand for product follows a Uniform Distribution or the Beta Distribution.

Our research results could be helpful for analysis of the quantity discount policy for the human-learning stochastic inventory model, supply chain coordination under worker learning and forgetting, and the analysis for the periodic review system for similar problem settings.

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APPENDIX A

Figure 1. Shapes and Characteristics of Functions $G''(q)$, $G'(q)$, and $G(q)$.